

# Digital

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## \* Steady State error (s.s.e)

$e(t) \rightarrow$  s.s.e: the error at  $t \rightarrow \infty$

$$S.s.e = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} s E(s)$$

$$= \lim_{z \rightarrow 1} (1-z^{-1}) E(z) = \lim_{z \rightarrow 1} (z-1) E(z)$$

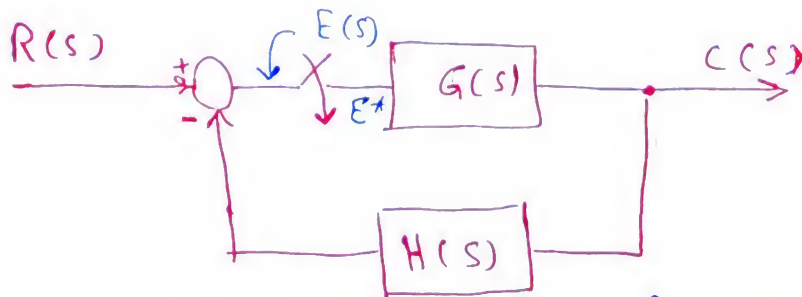
### # Cont. time

①  $r(t) = 1$  (Unit Step)  $\leftrightarrow$  s.s.e =  $\frac{1}{1+K_p} \Rightarrow K_p = \lim_{s \rightarrow 0} G H(s)$  [Position error constant]

②  $r(t) = t$  (Unit ramp)  $\Rightarrow$  s.s.e =  $\frac{1}{K_v} \Rightarrow K_v = \lim_{s \rightarrow 0} s G H(s)$  [Velocity error constant]

③  $r(t) = t^2/2$   $\Rightarrow$  s.s.e =  $\frac{1}{K_a} \Rightarrow K_a = \lim_{s \rightarrow 0} s^2 G H(s)$  [acceleration error constant]

## \* Steady-State error in discrete-time System



The general form for discrete-time feedback

$$S.s.e = \lim_{s \rightarrow 0} s E(s) = \lim_{z \rightarrow 1} (z-1) E(z)$$

? in terms of  $R(z)$

$$C(s) = G E^*$$

$$E(s) = R(s) - G H E^* \rightarrow \text{Starting } E^* = R^* - \overline{G H}^* E^*$$

$$(1 + \overline{G H}^*) E^* = R^*$$

$$E^*(s) = \frac{R^*(s)}{1 + \overline{G H}(s)} \Rightarrow E(z) = \frac{R(z)}{1 + \overline{G H}(z)}$$

$$\therefore \text{S.S.E} = \lim_{z \rightarrow 1} (z-1) \frac{R(z)}{1 + \overline{GH}(z)}$$

↓ depend on

① System i/p  $\Rightarrow R(z)$

② Digital O.L.T.F =  $\overline{GH}(z)$  or system type

# The general form for Digital O.L.T.F

$$\overline{GH}(z) = \frac{N(z)}{(z-1)^n D(z)}$$

where  $n \rightarrow$  system type

OR 
$$\overline{GH}(z) = \frac{N(z)}{(1-z^{-1})^n D(z)}$$

where  $n \rightarrow$  system type

\* For unit step input

$$r(t) = 1 \Rightarrow r(KT) = 1 \Rightarrow \boxed{R(z) = \frac{z}{z-1}}$$

$$\text{S.S.E} = \lim_{z \rightarrow 1} (z-1) \frac{R(z)}{1 + \overline{GH}(z)}$$

$$= \lim_{z \rightarrow 1} (\cancel{z-1}) \frac{z}{(\cancel{z-1})(1 + \overline{GH}(z))} = \lim_{z \rightarrow 1} \frac{z}{1 + \overline{GH}(z)}$$

$$\text{S.S.E} = \frac{1}{1 + \lim_{z \rightarrow 1} \overline{GH}(z)}$$

$$= \frac{1}{1 + K_p}$$

$$K_p = \lim_{z \rightarrow 1} \overline{GH}(z) \equiv \text{Position Error Constant}$$

$\Rightarrow$  continue

\* For unit ramp input

$$r(t) = t \Rightarrow r(KT) = KT \Rightarrow \left| R(z) = \frac{Tz}{(z-1)^2} \right|$$

$$S.S.E = \lim_{z \rightarrow 1} (z-1) \frac{R(z)}{1 + \overline{GH}(z)}$$

$$= \lim_{z \rightarrow 1} \cancel{(z-1)} \frac{Tz}{(z-1)^2 (1 + \overline{GH}(z))}$$

$$\begin{aligned} S.S.E &= \lim_{z \rightarrow 1} \frac{Tz}{(z-1) + (z-1) \overline{GH}(z)} \\ &= \frac{1}{0 + \lim_{z \rightarrow 1} (z-1) \overline{GH}(z)} \\ &= \frac{1}{\frac{1}{T} \lim_{z \rightarrow 1} (z-1) \overline{GH}(z)} = \frac{1}{K_v} \end{aligned}$$

$$K_v = \frac{1}{T} \lim_{z \rightarrow 1} (z-1) \overline{GH}(z) \equiv \text{Velocity error constant}$$

\* For acceleration input

$$r(t) = t^2 ; \quad t \xrightarrow{z \cdot T} \frac{Tz}{(z-1)^2}$$

$$\begin{aligned} R(z) &= Z \left( t \cdot f(t) \Big|_{\substack{\downarrow \\ t=KT}} \right) = -Tz \frac{df(z)}{dz} \\ &= Z(t, t) = -Tz \frac{d}{dz} \left( \frac{Tz}{(z-1)^2} \right) \\ &= -T^2 z \left[ \frac{(z-1)^2 - z(z-1)}{(z-1)^3} \right] \\ &= -T^2 z \left[ \frac{(z-1) - z}{(z-1)^3} \right] \end{aligned}$$

$$\left| R(z) = \frac{T^2 z(z+1)}{(z-1)^3} \right|$$

$\Rightarrow$  Continue

$$\text{So, } t^2 \xrightarrow{z.T} \frac{T^2 z(z+1)}{(z-1)^3}$$

$$r(t) = t^2/2 \Rightarrow R(z) = \frac{T^2/2 z(z+1)}{(z-1)^3}$$

$$\text{S.S.e} = \lim_{z \rightarrow 1} (z-1) \frac{R(z)}{1 + \overline{GH}(z)}$$

$$\text{S.S.e} = \lim_{z \rightarrow 1} \cancel{(z-1)} \frac{T^2/2 z(z+1)}{(z-1)^3 z (1 + \overline{GH}(z))}$$

$$= \lim_{z \rightarrow 1} \frac{T^2/2 z(z+1)}{(z-1)^2 + (z-1)^2 \overline{GH}(z)}$$

$$= \frac{T^2}{0 + \lim_{z \rightarrow 1} (z-1)^2 \overline{GH}(z)}$$

$$\text{S.S.e} = \frac{1}{\frac{1}{T^2} \lim_{z \rightarrow 1} (z-1)^2 \overline{GH}(z)} = \frac{1}{K_a}$$

$$K_a = \frac{1}{T^2} \lim_{z \rightarrow 1} (z-1)^2 \overline{GH}(z) = \text{acceleration constant}$$

\* Summary :

$$\textcircled{1} r(t) \Big|_{t=KT} = 1 \rightarrow \text{S.S.e} = \frac{1}{1 + K_p} \iff K_p = \lim_{z \rightarrow 1} \overline{GH}(z)$$

$$\textcircled{2} r(t) = t \Big|_{t=KT} \rightarrow \text{S.S.e} = \frac{1}{K_v} \iff K_v = \frac{1}{T} \lim_{z \rightarrow 1} (z-1) \overline{GH}(z)$$

$$\textcircled{3} r(t) = t^2/2 \Big|_{t=KT} \rightarrow \text{S.S.e} = \frac{1}{K_a} \iff K_a = \frac{1}{T^2} \lim_{z \rightarrow 1} (z-1)^2 \overline{GH}(z)$$



IP type	Unit step $r(t)=1$	Unit ramp $r(t)=t$	Acceleration i/p $r(t)=t^2/2$
0	$\frac{1}{1+K_p}$	$\infty$	$\infty$
1	0	$1/K_v$	$\infty$
2	0	0	$1/K_a$

Ex 1:-

$$\textcircled{1} \quad \overline{GH}(z) = \frac{-10}{(z-1)(1-z)(z^2-1.4z+1)} = \frac{10}{(z-1)^2(z^2-1.4z+1)}$$

$$\textcircled{2} \quad \overline{GH}(z) = \frac{(z+1)}{(z-1)(z-z)}$$

$$\textcircled{3} \quad \overline{GH}(z) = \frac{z+0.5}{(1-z)^2(z^2-z)}$$

Find: (i) System type and order

(ii) error constants

(iii) s.s.e for  $r(t)=1$ ;  $r(t)=t$ ;  $r(t)=2+3t+4t^2$

$\boxed{1}$  type=2 & order = 4 (4th order system)

$$K_p = \lim_{z \rightarrow 1} \overline{GH}(z) = \infty$$

$$K_v = \frac{1}{T} \lim_{z \rightarrow 1} (z-1) \overline{GH}(z) \quad (\text{assume } T=1s)$$

$$= \infty$$

$$K_a = \frac{1}{T^2} \lim_{z \rightarrow 1} (z-1)^2 \overline{GH}(z) = \frac{1}{1} \frac{10}{1-1.4+1} = \frac{10}{0.6}$$

$\longrightarrow$  continue

$$r(t) = 1 \rightarrow \text{s.s.e} = \frac{1}{1+K_p} = \frac{1}{\infty} = 0$$

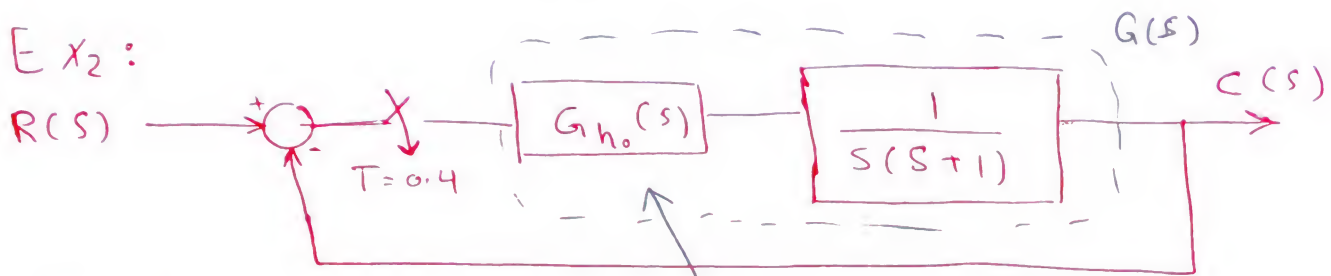
$$r(t) = t \rightarrow \text{s.s.e} = \frac{1}{K_v} = \frac{1}{\infty} = 0$$

$$r(t) = 2 + 3t + 4t^2 \xrightarrow{K_v} \text{s.s.e} = 2 \left( \frac{1}{1+K_p} \right) + 3 \left( \frac{1}{K_v} \right) + 8 \left( \frac{1}{K_a} \right)$$

$\downarrow$   $\frac{8t^2}{2}$

$$\therefore \text{s.s.e} = 8 \left( \frac{0.6}{1.0} \right) = 0.48$$

Ex<sub>2</sub>:



Find

$$z.o.h = \frac{1-e^{-Ts}}{s}$$

- ① O.L. pulse T.F.
- ② C.L. pulse T.F.
- ③ type and order of the system
- ④ S.S.e for unit step and unit ramp inputs
- ⑤ steady state value of the o/p for unit step i/p
- ⑥ find the first 3 terms of o/p response  
 $y(0), y(0.4), y(0.8)$  for unit step i/p

$$\text{① O.L.T.F} = \overline{G H(z)} \Rightarrow G(z)$$

$$\text{C.L.T.F} = \frac{G(z)}{1 + \overline{G H(z)}} \Rightarrow \frac{G(z)}{1 + G(z)}$$

$\text{o.l.t.f}$

$$\text{O.L.T.F} = G(z) = z \left[ \frac{1-e^{-Ts}}{s^2(s+1)} \right]$$

$$= (1-z^{-1}) Z \left[ \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+1} \right]$$

$$[A=1], [C=1], B \parallel \frac{1}{2} = A + B + \frac{C}{2} = 1 + B + \frac{1}{2} \Rightarrow [B = -1]$$

$\Rightarrow$  continue

$$\begin{aligned}
 \text{O.L.T.F} &= (1 - z^{-1}) z \left[ \underset{\substack{\uparrow \\ 1/s^2}}{t} + \underset{\substack{\uparrow \\ 1/s}}{(-1)} + \underset{\substack{\uparrow \\ 1/s+1}}{e^{-t}} \right], \quad t = kT \\
 &= (1 - z^{-1}) \left[ \frac{Tz}{(z-1)^2} - \frac{z}{(z-1)} + \frac{z}{z - e^{-T}} \right] \\
 &= \left( \frac{z-1}{z} \right) \left[ \frac{0.4z}{(z-1)^2} - \frac{z}{(z-1)} + \frac{z}{z-0.67} \right] \quad \left| \begin{array}{l} e^{-T} = e^{-0.4} \\ = 0.67 \end{array} \right. \\
 &= \frac{0.4}{(z-1)} - 1 + \frac{z-1}{z-0.67} \\
 &= \frac{0.4(z-0.67) - (z-1)(z-0.67) + (z-1)^2}{(z-1)(z-0.67)(z-1)}
 \end{aligned}$$

$$\boxed{G(z) = \frac{0.07z + 0.062}{(z-1)(z-0.67)}} \quad \begin{array}{l} \text{open loop pulse T.F.} \\ \text{(O.L. Digital T.F.)} \end{array}$$

note: in general, O.L.T.F. =  $GH(z)$ ; since  $H(z) = 1$

$$\therefore \text{O.L.T.F.} = G(z)$$

$$\begin{aligned}
 \textcircled{2} \text{ C.L.T.F} &= \frac{G(z)}{1 + GH(z)} = \frac{G(z)}{1 + G(z)} \\
 &= \frac{\frac{0.07z + 0.062}{(z-1)(z-0.67)}}{1 + \frac{0.07z + 0.062}{(z-1)(z-0.67)}} = \frac{0.07z + 0.062}{(z-1)(z-0.67) + 0.07z + 0.062}
 \end{aligned}$$

$\Rightarrow$  continue

③ type : 1

based on o.l.T.F.

order: 2 "second order system" based on o.l.T.F.

④ s.s.e for unit step and unit ramp

$$K_p = \lim_{z \rightarrow 1} G H(z) = \lim_{z \rightarrow 1} G(z) = \infty$$

$$\text{s.s.e} \Big|_{\substack{r(t)=1 \\ \text{unit step}}} = \frac{1}{1+K_p} = \frac{1}{\infty} = 0$$

$$K_v = \frac{1}{T} \lim_{z \rightarrow 1} (z-1) \overline{G H(z)}$$

$$= \frac{1}{0.4} \lim_{z \rightarrow 1} (z-1) G(z) = \frac{1}{0.4} \left[ \frac{0.07 + 0.062}{1 - 0.67} \right] = 1$$

$$\Rightarrow \text{s.s.e} = \frac{1}{K_v} = 1$$

⑤  $C(\infty) \equiv \text{final value} \equiv \text{steady state value of output}$

$$= \lim_{z \rightarrow 1} (z-1) C(z)$$

$$\text{C.L.T.F} = \frac{C(z)}{R(z)} = \frac{G(z)}{1 + G H(z)} = \left( \frac{0.07z + 0.062}{z^2 - 1.6z + 0.732} \right)$$

$$C(z) = \text{T.F} \times R(z)$$

$$= \left[ \frac{0.07z + 0.062}{z^2 - 1.6z + 0.732} \right] R(z)$$

$$\text{for unit step i/p } (R(z) = \frac{z}{z-1})$$

$$C(z) = \frac{(0.07z + 0.062)z}{(z-1)(z^2 - 1.6z + 0.732)}$$



$$C(\infty) = \lim_{z \rightarrow 1} \frac{(0.07z + 0.062)z}{z^2 - 1.6z + 0.732}$$

$$= \frac{0.07 + 0.062}{1 - 1.6 + 0.732} = 1$$

# Another method to get  $C(\infty)$

$$e(t) = r(t) - y(t)$$

$$e(\infty) = r(\infty) - y(\infty)$$

$\downarrow$  s.s.e       $\downarrow$  s.s.e       $\downarrow$  steady state o/p

$$y(\infty) = r(\infty) - e(\infty) \quad ; \text{ for unit step input}$$

$\Downarrow$  1       $\downarrow$  0

$$\therefore \boxed{y(\infty) = 1 = C(\infty)}$$

⑥ Find  $C(0)$ ;  $C(0.4)$ ;  $C(0.8)$

from requirement (5) we have

$$C(z) = (T, F) R(z)$$

$$= \left[ \frac{0.07z + 0.062}{z^2 - 1.6z + 0.732} \right] \left( \frac{z}{z-1} \right)$$

$$= \frac{0.07z^2 + 0.062z}{z^3 - 2.6z^2 + 2.33z - 0.732}$$

we use long division to obtain

first 3 terms

$$\begin{array}{r}
 0.07z^{-1} \\
 \hline
 z^3 - 2.6z^2 + 2.33z - 0.732 \quad \begin{array}{l} 0.07z^2 + 0.062z \\ 0.07z^2 + 0.182z + 0.1631 + 0.0512z^{-1} \end{array} \\
 \hline
 0.244z - 0.163 + 0.512z^{-1}
 \end{array}$$

$$C(z) = 0.07z^{-1} + 0.244z^{-2} + \dots$$

$$= \underset{y_0}{C(0)} + C(T)z^{-1} + C(2T)z^{-2} + \dots$$

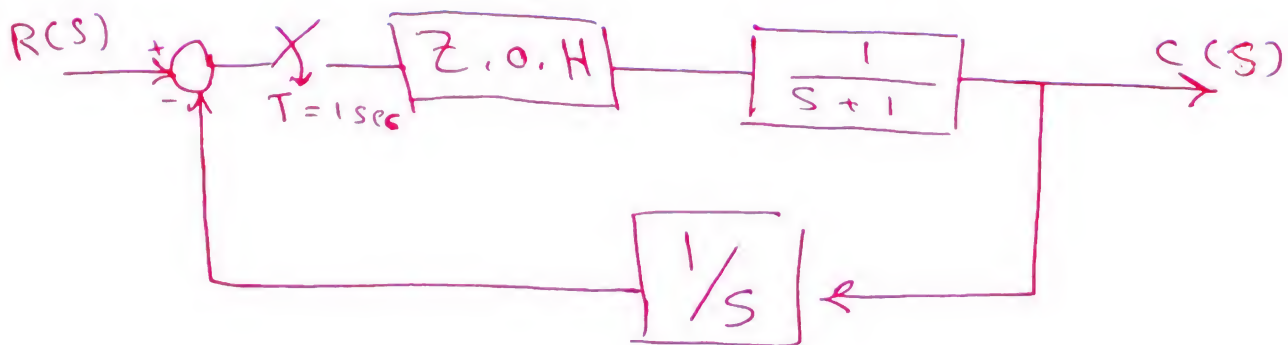
$$= C(0) + C(0.4)z^{-1} + C(0.8)z^{-2} + \dots$$

$$C(0) = 0$$

$$C(T) = C(0.4) = 0.07$$

$$C(2T) = C(0.8) = 0.244$$

⇒ Report ⇐



Find the Unit-Step response  
and discuss System Stability

# Relation between Z-domain and S-domain

$$Z = e^{TS}$$

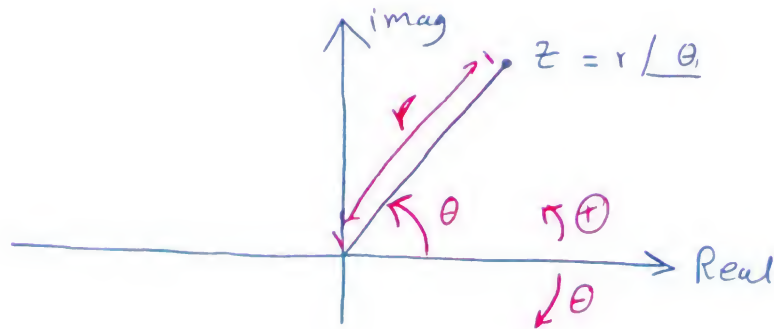
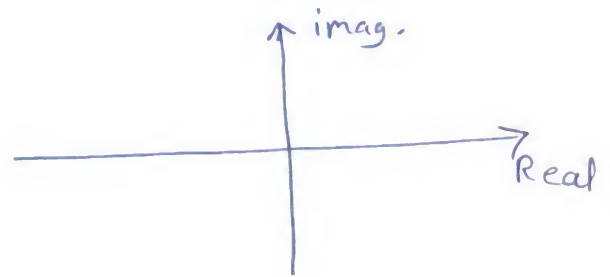
for  $S = \alpha + j\omega$   
 Real  $\leftarrow \alpha$   $\leftarrow$  imag  $\leftarrow \omega$

$$Z = e^{T(\alpha + j\omega)}$$

$$Z = e^{\tilde{r}T} \cdot e^{j\omega T} = \underbrace{1}_{\text{mag}} \underbrace{1}_{\text{angle}}$$

$$= r \angle \theta$$

$$= e^{\tilde{r}T} \angle \omega T \Rightarrow T=1\text{sec} \Rightarrow r = e^{\tilde{r}}, \theta = \omega$$



for  $S_0 = 0 + j0 \Rightarrow r = e^0 \angle 0^\circ = 1 \angle 0^\circ$

$\therefore S_1 = j\pi/4 \Rightarrow r = e^0 \angle \pi/4 = 1 \angle \pi/4 = 45^\circ$

$\therefore S_2 = j\pi/2 \Rightarrow r = e^0 \angle \pi/2 = 1 \angle 90^\circ$

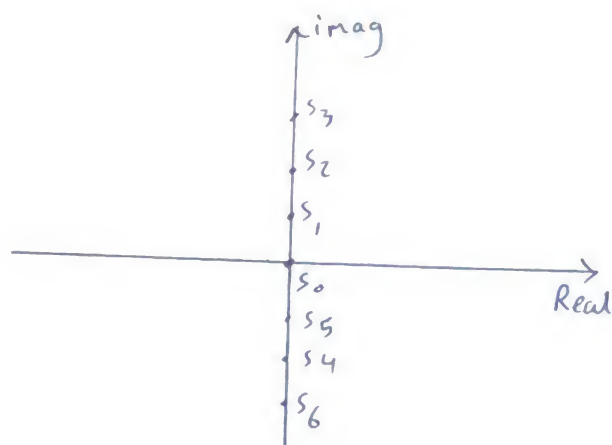
$\therefore S_3 = j\pi \Rightarrow r = e^0 \angle \pi = 1 \angle 180^\circ$

$\therefore S_4 = -j\pi/2 \Rightarrow r = e^0 \angle -\pi/2 = 1 \angle -90^\circ$

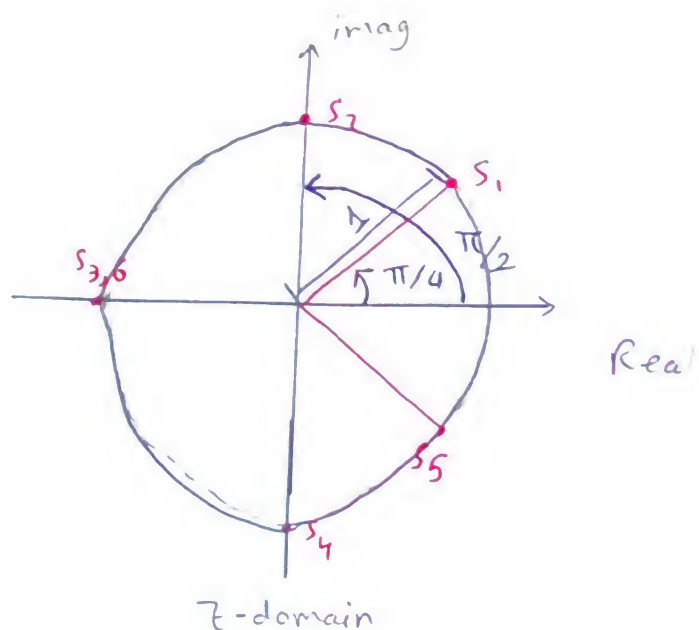
$\therefore S_5 = -j\pi/4 \Rightarrow r = e^0 \angle -\pi/4 = 1 \angle -45^\circ$

$\therefore S_6 = -j\pi \Rightarrow r = e^0 \angle -\pi = 1 \angle -180^\circ$

$\Rightarrow$  continue



S-domain



Z-domain

Mapping the imag. axis from S-domain to Z-domain is a circle with radius  $r=1$  and center at origin

- For LHS in S-domain ( $S = \alpha + j\omega$ )

$$e^{\alpha} (\alpha < 0 \text{ (-ve)})$$

$$e^{\alpha} < 1 \Rightarrow |z| = r = |e^{\alpha}| < 1 \text{ (stable)}$$

(Pole lies in unit circle; stable pole)

- For RHS in S-domain ( $S = \alpha + j\omega$ )

$$e^{\alpha} (\alpha > 0 \text{ (+ve)})$$

$$e^{\alpha} > 1 \Rightarrow |z| = r = |e^{\alpha}| > 1 \text{ (unstable)}$$

(Pole lies outside of unit circle; unstable pole)

- For imag axis

$$\alpha = 0 \Rightarrow e^{\alpha=0} = 1 \text{ (critical Pole)}$$

$$z = r = e^{\alpha} = 1 \text{ (critical Pole)}$$

(Pole lies on unit circle; critical pole)

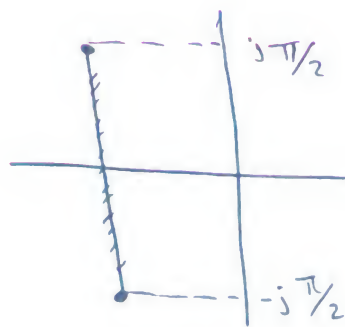


1] The System is stable if all poles lie inside unit circle ( $|z_p| < 1$ )

2] The system is unstable if at least one pole lies outside unit circle ( $|z| > 1$ )

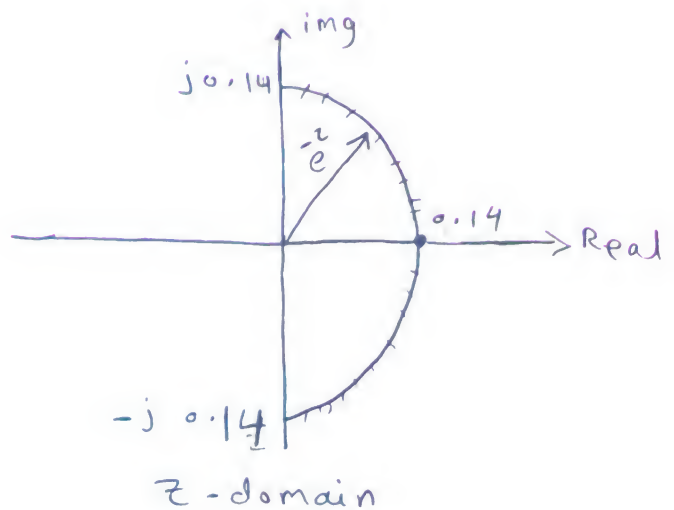
3] The system is critical stable, if one or more pole lie on the unit circle and the other poles lie inside the unit circle

Example: map to z-domain

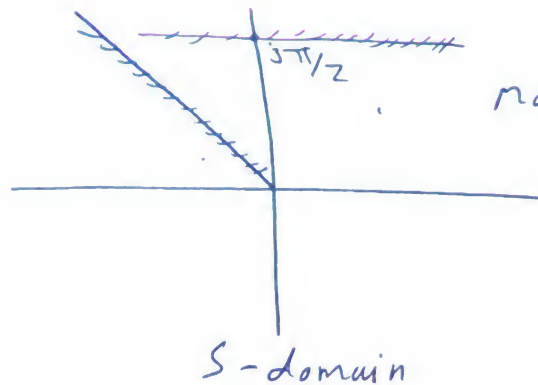


s-domain

$\Rightarrow$



Report



map to z-domain